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Existence of positive solutions of singular elliptic boundary value problems in a ball

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ABSTRACT

This paper deals with existence theorems of positive solutions for singular elliptic boundary value problems in a ball. The results of this paper are generalizations of those proved by authors cited in the references.

1. Introduction

Considerable attention has been given to the existence of positive solutions for singular elliptic boundary value problems by several authors including Grandall [1] and Usmani [2]. In particular, the following singular elliptic boundary value problems have been studied:

$$\text{(i) } \Delta u + p(x)u^{-\lambda} = 0, \quad x \in B \subset R^N, \tag{1.1}$$

$$u = 0, \quad x \in \partial B, \tag{1.2}$$

and

$$\text{(ii) } \Delta u + f(x, u) = 0, \quad x \in B \subset R^N, \tag{1.3}$$

$$u = 0, \quad x \in \partial B, \tag{1.4}$$

where Δ is an N-dimensional Laplacian ($N \geq 2$), $x = (x_1, x_2, \ldots, x_N)$ is a vector, B is an open ball centered at the origin of R^N, that is, $B = \{x \in R^N \mid |x| < 1\}$, ∂B is the boundary of B, $\lambda > 0$, and $f(x, u)$ is singular in x and u.

In spite of these studies of the above problems, hardly any attention is given to the existence of positive solutions of the following singular boundary value problem

$$\Delta u + f(x, \nabla u)u^{-\lambda} = 0, \quad x \in B \subset R^N, \tag{1.5}$$

$$u = 0, \quad x \in \partial B, \tag{1.6}$$

there ∇ denotes the N-dimensional gradient operator.

Recently, Xu and Debnath [3–6] have considered the existence of positive entire solutions of singular boundary value problems, in general, and singular nonlinear polyharmonic equations, in particular.

The main purpose of this paper is to prove the existence of a positive solution $u(x) \in C^2(B) \cap C(\overline{B})$ for boundary value problems (1.5)–(1.6). The results of this paper are generalizations of those in [1,2].

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2. Several results and lemmas

We consider that the function f is radially symmetric, that is

$$f = F(|x|, |\nabla u|).$$

Thus, we discuss the existence of the positive solutions for a singular ordinary differential equation boundary value problem

$$(t^{N-1}y')' + t^{N-1}F(t, |y'|)y^{-\lambda} = 0, \quad 0 < t < 1, \tag{2.1}$$

$$y'(0) = 0, \qquad y'(1) = 0. \tag{2.2}$$

We assume that the function $F(t, z)$ satisfies the following conditions (A_1), (A_2) and (A_3):

(A_1). $F : [0, 1) \times [0, \infty) \to (0, \infty)$ is continuous, $F(t, z)$ is continuously differentiable in $z \geq 0$. For each fixed $t \in [0, 1)$, $F(t, z)$ is strictly increasing in $z \geq 0$;

(A_2). There exists a positive constant $M > 0$ satisfying

$$\int_0^1 \int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, d)\mathrm{d}r\mathrm{d}s < M,$$

which holds uniformly for every $d \geq 0$.

(A_3). For all $\varepsilon > 0$, there exists $\delta > 0$, such that $1 - \delta < t < 1$, the following inequality

$$\int_0^1 \int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, d)\mathrm{d}r\mathrm{d}s \leq \varepsilon,$$

holds uniformly for every $d \geq 0$.

We study the existence of positive solutions of boundary value problems (2.1)–(2.2) by the shooting method. We consider that the unique positive solution of the initial value problem

$$(t^{N-1}y')' + t^{N-1}F(t, |y'|)y^{-\lambda} = 0, \tag{2.3}$$

$$y(0) = \alpha, \qquad y'(0) = 0, \tag{2.4}$$

where $\alpha > 0$ is a parameter. Condition (A_1) implies that problems (2.3)–(2.4) have a unique positive solution $y_\alpha(t)$, $t \in [0, T_\alpha)$, where $[0, T_\alpha)$ is the maximal existence interval in $y_\alpha(t)$ (see [7, p. 1164]). Clearly, the value of T_α lies in $0 < T_\alpha \leq 1$. If $T_\alpha < 1$, then $y_\alpha(t) > 0$, $0 \leq t < T_\alpha$; $y_\alpha(T_\alpha) = 0$. Therefore, $y_\alpha(t)$ depends continuously on its initial value α.

Lemma 2.1. *Suppose that F satisfies conditions (A_1), (A_2) and (A_3). Let α and β be positive numbers satisfying $\alpha > \beta$. If $y_\beta(t)$ exists on $[0, T)$, $(0 \leq T < 1)$, then $y_\alpha(t)$ also exists on $[0, T)$ and satisfies*

$$y_\alpha(t) = y_\beta(t), \quad t \in [0, T), \tag{2.5}$$

$$y'_\alpha(t) = y'_\beta(t), \quad t \in (0, T). \tag{2.6}$$

Proof. We first prove Lemma 2.1 in three steps.

(i). Assume that $y_\alpha(t)$ and $y_\beta(t)$ are defined on $[0, T)$. Then, $y'_\alpha(t) > y'_\beta(t)$, $t \in [0, T)$. In fact, (2.3)–(2.4) and (A_1) can be applied to get

$$y'_\alpha(t) = -\int_0^t \left(\frac{s}{t}\right)^{N-1} F(s, |y'_\alpha(s)|)y_\alpha^{-\lambda}(s)\mathrm{d}s < 0.$$

Hence $y'_\alpha(t)$ is strictly decreasing and $|y'_\alpha(t)| = -y'_\alpha(t)$. Similarly, $y'_\beta(t)$ is also strictly decreasing and $|y'_\beta(t)| = -y'_\beta(t)$, we find

$$y'_\alpha(t) - y'_\beta(t) = -\int_0^t \left(\frac{s}{t}\right)^{N-1} [F(s, |y'_\beta(s)|)y_\beta^{-\lambda}(s) - F(s, |y'_\alpha(s)|)y_\alpha^{-\lambda}(s)]\mathrm{d}s, \quad t \in [0, T). \tag{2.7}$$

Choose a positive number γ such that $\beta < \gamma < \alpha$. Since $y_\alpha(t)$ is continuous and $y_\alpha(0) = \alpha$, $t \in [0, T)$, there exists $t_0 > 0 (0 < t_0 < T)$ satisfying $y_\alpha(t) > \gamma$, $0 \leq t \leq t_0$, and

$$F(t, |y'_\beta(t)|)y_\beta^{-\lambda}(t) - F(t, |y'_\alpha(t)|)y_\alpha^{-\lambda}(t) > F(t, 0)\beta^{-\lambda} - F(t, |y'_\alpha(t)|)\gamma^{-\lambda},$$
$$0 \leq t \leq t_0. \tag{2.8}$$

Let $F(0, 0)\beta^{-\lambda} - F(0, 0)\gamma^{-\lambda} = a$. Since $F(t, |y'_\alpha(t)|)y_\alpha^{-\lambda}(t)$ is continuous and $y'_\alpha(0) = y'_\beta(0) = 0$, there exists $\delta \in [0, t_0)$ such that

$$F(t, |y'_\alpha(t)|)\gamma^{-\lambda} < F(0, 0)\gamma^{-\lambda} + \frac{\alpha}{2}, \tag{2.9a}$$

$$F(t, 0)\beta^{-\lambda} > F(0, 0)\beta^{-\lambda} - \frac{\alpha}{2}, \tag{2.9b}$$

where $0 < t < \delta$. It follows from (2.8), (2.9a) and (2.9b) that

$$F(t, |y'_\beta(t)|)y_\beta^{-\lambda}(t) - F(t, |y'_\alpha(t)|)y_\alpha^{-\lambda}(t) > 0, \quad 0 < t < \delta \leq T. \tag{2.10}$$

It also follows from (2.7) and (2.10), that $y'_\alpha(t) > y'_\beta(t)$, $0 < t < \delta \leq T$. Now, we prove that $\delta = T$.

If $\delta < T$, we find that $\delta_1 : \delta < \delta_1 < T$, such that

$$y'_\alpha(t) > y'_\beta(t), \quad 0 < t < \delta_1, \qquad y'_\alpha(\delta_1) = y'_\beta(\delta_1). \tag{2.11}$$

Note that $y_\alpha(0) - y_\beta(0) = \alpha - \beta$, $y'_\alpha(t) \leq 0$, $y'_\beta(t) \leq 0$, $0 < t < \delta_1$. Thus,

$$y_\alpha(t) > y_\beta(t), \qquad |y'_\alpha(t)| < |y'_\beta(t)|, \qquad 0 < t < \delta_1. \tag{2.12}$$

Hence, we have

$$F(t, |y'_\beta(t)|)y_\beta^{-\lambda}(t) > F(t, |y'_\alpha(t)|)y_\alpha^{-\lambda}(t), \quad 0 < t < \delta_1,$$

and

$$\begin{aligned} 0 &= y'_\alpha(\delta_1) - y'_\beta(\delta_1) \\ &= \int_0^{\delta_1} \left(\frac{s}{\delta_1}\right)^{N-1} [F(s, |y'_\beta(s)|)y_\beta^{-\lambda}(s) - F(s, |y'_\alpha(s)|)y_\alpha^{-\lambda}(s)]\mathrm{d}s > 0. \end{aligned}$$

This contradiction proves that $y'_\alpha(t) > y'_\beta(t)$, $0 < t < T$, that is (2.6).

(ii). If $y_\beta(t)$ exists on $[0, T)$, then $y_\alpha(t)$ also exists on $[0, T)$.

In fact, we assume that the existence interval of $y_\alpha(t)$ is less that $[0, T)$ (that is $y_\alpha(t)$ vanishes before $t = T$). Since $y_\alpha(t) > y_\beta(t)$ near the origin, the curve $y_\alpha(t)$ is sure to intersect the curve $y_\beta(t)$. Suppose that the first intersection point is $t = \tau < T$. Then we have

$$y_\alpha(t) > y_\beta(t), \quad 0 \leq t < \tau; \qquad y_\alpha(\tau) = y_\beta(\tau).$$

Using (2.3), (2.4), condition (A_1) and the conclusion of (i), we obtain

$$-y_\alpha(\tau) + \alpha + y_\beta(t) - \beta = \int_0^\tau \int_0^s \left(\frac{r}{s}\right)^{N-1} [F(r, |y'_\alpha(r)|)y_\alpha^{-\lambda}(r) - F(r, |y'_\beta(r)|)y_\beta^{-\lambda}(r)]\mathrm{d}r\mathrm{d}s \leq 0.$$

Thus, $y_\beta(\tau) - y_\alpha(\tau) \leq \beta - \alpha < 0$. This contradiction proves that $y_\alpha(t)$ also exists on $[0, T)$.

(iii). We prove that (2.5) holds. In fact, making use of $y_\alpha(0) - y_\beta(0) = \alpha - \beta > 0$ and the conclusions of (i), (ii), we can prove immediately that (2.5) holds. Lemma 2.1 is proved. □

Lemma 2.2. *Under the assumptions (A_1), (A_2) and (A_3), the boundary value problems (2.1)–(2.2) have a unique positive solution $y \in C^2[0, 1) \cap C[0, 1]$.*

Proof. Define the subsets $\overline{S}$, $\underline{S} \subset (0, \infty)$, respectively, by

$$\begin{aligned} \overline{S} &= \{\alpha > 0 | y_\alpha(t) \text{ exists on } [0, 1) \text{ and satisfies } y_\alpha(1) > 0\}; \\ \underline{S} &= \{\alpha > 0 | y_\alpha(t) \text{ vanishes before } t = 1\}. \end{aligned}$$

It follows from Lemma 2.1 that for all $\alpha \in \overline{S}$ and for all $\beta \in \underline{S}$, $\alpha > \beta$. Thus, $\overline{S} \cap \underline{S} = \oslash$. The following results (i)–(v) are valid.

(i). $\overline{S}$ is not empty.

Choose arbitrarily a positive number α_1 such that $\frac{\alpha_1}{2} > M > 1$. Thus, by condition (A_2), α_1 satisfies

$$\int_0^1 \int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, d) \left(\frac{\alpha_1}{2}\right)^{-\lambda} \mathrm{d}r\mathrm{d}s < \int_0^1 \int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, d)\mathrm{d}r\mathrm{d}s < M < \frac{\alpha_1}{2}, \tag{2.13}$$

which holds uniformly for every $d \geq 0$. We claim that $y_{\alpha_1}(t) > \frac{\alpha_1}{2}$, for $t \in [0, 1)$. In fact, if this is not the case, then there exists $t_1 \in (0, 1)$ such that

$$y_{\alpha_1}(t) > \frac{\alpha_1}{2}, \quad t \in [0, t_1); \qquad y_{\alpha_1}(t_1) = \frac{\alpha_1}{2}. \tag{2.14}$$

Making use of (2.3), (2.4), we get

$$y_{\alpha_1}(t) - \alpha_1 + \int_0^{t_1} \int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, |y'_{\alpha_1}(r)|)y_{\alpha_1}^{-\lambda}(r)\mathrm{d}r\mathrm{d}s = 0.$$

Let $d_1 = \max_{0\le r\le t} |y'_{\alpha_1}(r)|$. Eqs. (2.13), (2.14) and condition (A_1) can be applied to get

$$\begin{aligned}\frac{\alpha_1}{2} &= \int_0^{t_1}\int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, |y'_{\alpha_1}(r)|) y_{\alpha_1}^{-\lambda}(r)\mathrm{d}r\mathrm{d}s \\ &\le \int_0^{1}\int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, d_1)\left(\frac{\alpha_1}{2}\right)^{-\lambda}\mathrm{d}r\mathrm{d}s < \frac{\alpha_1}{2}.\end{aligned}$$

This contradiction implies $y_{\alpha_1}(t) > \frac{\alpha_1}{2}$, $t \in [0, 1)$. Thus, $\alpha_1 \in \overline{S}$, i.e. $\overline{S}$ is not empty.

(ii). $\underline{S}$ is not empty.

Let $\int_0^{\frac{1}{2}}\int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, 0)\mathrm{d}r\mathrm{d}s = k$, ($k > 0$ is a constant number). Choose arbitrarily a positive number α' such that $\alpha' < \min\{k, 1\}$, we have

$$\int_0^{\frac{1}{2}}\int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, 0)(\alpha')^{-\lambda}\mathrm{d}r\mathrm{d}s > k > \alpha'. \tag{2.15}$$

Then, for such an α', $y_{\alpha'}(t)$ must vanish before $t = \frac{1}{2}$. In fact, if $y_{\alpha'}(t)$ can be prolonged to $t = \frac{1}{2}$ and remains positive. Since $y_{\alpha'}(t) \le \alpha'$, $t \in \left[0, \frac{1}{2}\right]$, making use of (2.3), (2.4) and (2.15) gives

$$\begin{aligned}-y_{\alpha'}\left(\frac{1}{2}\right) &= -\alpha' + \int_0^{\frac{1}{2}}\int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, |y'_{\alpha'}(r)|) y_{\alpha'}^{-\lambda}(r)\mathrm{d}r\mathrm{d}s \\ &\ge -\alpha' + \int_0^{\frac{1}{2}}\int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, 0)(\alpha')^{-\lambda}\mathrm{d}r\mathrm{d}s \\ &> -\alpha' + \alpha' = 0.\end{aligned}$$

This contradiction implies $\alpha' \in \underline{S}$.

(iii). $\inf \overline{S}$ does not belong to $\overline{S}$.

Put $\alpha_* = \inf \overline{S}$, it is clear that $\alpha_* \in (0, \infty)$. Suppose that $\alpha_* \in \overline{S}$, then $y_{\alpha_*}(1) \equiv l > 0$. Using condition (A_3), for $\left(\frac{l}{2}\right)^{\lambda+1} > 0$, there exists $\delta_1 > 0$ and choose $t_1 \in (0, 1)$ sufficiently close to 1, satisfying $1 - \delta_1 < t_1 < 1$, so that

$$\int_{t_1}^{1}\int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, d)\left(\frac{l}{2}\right)^{-\lambda}\mathrm{d}r\mathrm{d}s < \left(\frac{l}{2}\right)^{\lambda+1}\cdot\left(\frac{l}{2}\right)^{-\lambda} = \frac{l}{2} \tag{2.16}$$

which holds uniformly for every $d \ge 0$. Since $y'_{\alpha_*}(t) < 0$, $(0 < t < 1)$, we get that $y_{\alpha_*}(t_1) > l$. Noting the continuous dependence of solutions of (2.3), (2.4) on initial data, for all $\alpha_0 \in (0, \alpha_*)$ sufficiently close to α_*, $y_{\alpha_0}(t)$ are define on $[0, t_1]$ and satisfy

$$y_{\alpha_0}(t_1) > l. \tag{2.17}$$

Now, we claim that such a $y_{\alpha_0}(t)$ satisfies $y_{\alpha_0}(t) > \frac{l}{2}$ on its interval of existence and, consequently, can be extended to $[0, 1)$. In fact if this is not true, then, there is $t_2 \in (t_1, 1)$ such that

$$y_{\alpha_0}(t) > \frac{l}{2}, \quad t_1 \le t < t_2; \qquad y_{\alpha_0}(t_2) = \frac{l}{2}. \tag{2.18}$$

Integrating (2.3) and making use of (2.18), (2.17) and (2.16), we obtain

$$\begin{aligned}\frac{l}{2} &= y_{\alpha_0}(t_2) = y_{\alpha_0}(t_1) - \int_{t_1}^{t_2}\int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, |y'_{\alpha_0}(r)|) y_{\alpha_0}^{-\lambda}(r)\mathrm{d}r\mathrm{d}s \\ &\ge l - \int_{t_1}^{1}\int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, d_0)\left(\frac{l}{2}\right)^{-\lambda}\mathrm{d}r\mathrm{d}s \\ &> l - \frac{l}{2} = \frac{l}{2},\end{aligned}$$

where $d_0 = \max_{t_1\le r\le t_2} |y'_{\alpha_0}(r)|$. Therefore, such an α_0 must be a member of $\overline{S}$. This contradicts the definition $\alpha_* = \inf \overline{S}$. Thus, $\inf \overline{S}$ does not belong to $\overline{S}$.

(iv). $\sup \underline{S}$ does not belong to $\underline{S}$.

Suppose that $\alpha^* = \sup \underline{S}$ and let t_1 be a point in $(0, 1)$ such that $y_{\alpha*}(t_1) = 0$. Choose $T \in (t_1, 1)$ arbitrarily and let it be fixed. Note that

$$\int_{t_1}^{T}\int_1^s \left(\frac{r}{s}\right)^{N-1} F(r, 0)\mathrm{d}r\mathrm{d}s > 0,$$

there exists $\varepsilon > 0$ sufficiently small such that

$$\int_{t_1}^{T}\int_{t_1}^{s}\left(\frac{r}{s}\right)^{N-1}F(r,0)\varepsilon^{-\lambda}\mathrm{d}r\mathrm{d}s > \varepsilon. \tag{2.19}$$

By using Lemma 2.1 and the continuous dependence of solutions on initial data, we find that $y_\beta(t)$ exists on $[0, t_1]$ and satisfies $0 < y_\beta(t_1) < \varepsilon$ for all $\beta > \alpha^*$ sufficiently close to α^*. Now, we assert that such a $y_\beta(t)$ vanishes before $t = T$. Assume on the contrary that $y_\beta(t)$ exists on $[0, T]$ and remains positive. Then we obtain that $0 < y_\beta(t) < \varepsilon, t_1 \le t \le T$, and integrating (2.3) twice and using (2.19), (A_1), we obtain

$$\begin{aligned}-y_\beta(T) &= -y_\beta(t_1) + \int_{t_1}^{T}\int_{0}^{s}\left(\frac{r}{s}\right)^{N-1}F(r,|y'_\beta(r)|)y_\beta^{-\lambda}(r)\mathrm{d}r\mathrm{d}s\\ &\ge -\varepsilon + \int_{t_1}^{T}\int_{0}^{s}\left(\frac{r}{s}\right)^{N-1}F(r,0)\varepsilon^{-\lambda}\mathrm{d}r\mathrm{d}s\\ &> -\varepsilon+\varepsilon = 0.\end{aligned}$$

This contradiction shows that such a β is contained in $\underline{S}$. However, this contradicts the definition of $\alpha^* = \sup \underline{S}$. Thus, $\sup \underline{S}$ does not belong to $\underline{S}$.

(v). $\alpha_0 \equiv \inf \underline{S} \equiv \sup \underline{S}$.

It is obvious that for all $\alpha \in \overline{S}$ and for all $\beta \in \underline{S}$, then $\beta < \alpha$. Thus, $\sup \underline{S} \le \inf \overline{S}$. Now, we claim that $\sup \underline{S} < \inf \overline{S}$ does not hold. In fact, if this is not true, choose $\alpha, \beta \in (\sup \underline{S}, \inf \overline{S})$ arbitrarily and assume $\alpha > \beta$. We see clearly that α, β belong neither to $\overline{S}$ nor to $\underline{S}$. We find that $y_\alpha(t), y_\beta(t)$ exist in $[0, 1)$, $y_\alpha(t) = y_\beta(t) = 0$ from the definition of $\overline{S}$ and $\underline{S}$. Since $\alpha > \beta$, by Lemma 2.1 we have

$$y'_\alpha(t) > y'_\beta(t), \quad t \in (0, 1).$$

Thus, $y_\alpha(t) - y_\beta(t)$ is strictly increasing in $[0, 1)$ and we have

$$y_\alpha(1) - y_\beta(1) > y_\alpha(0) - y_\beta(0) = \alpha - \beta > 0.$$

This contradiction proves that $\sup \underline{S} < \inf \overline{S}$ does not hold. Thus, $\alpha_0 \equiv \inf \overline{S} \equiv \sup \underline{S}$. It follows that $y_{\alpha_0}(t)$ is the unique positive solution of class $C^2[0, 1) \cap C[0, 1]$ of problems (2.1)–(2.2). This completes the proof of Lemma 2.2. □

3. Main results

We consider the singular elliptic boundary value problems (1.5)–(1.6) under the following conditions:

(B_1). $f : B \times R^N \to (0, \infty)$ is locally Hölder continuous with exponent $\theta \in (0, 1)$, and $f(x, p)$ is continuously differentiable in p. For every compact region $\Omega \subset B$, there exists an ordinary number ρ_Ω, such that $|f(x, p)| \le \rho_\Omega(1+|p|^2)$, $x \in \Omega$, $p \in R^N$;

(B_2). There exists functions $f_*, f^* : [0, 1) \times [0, \infty) \to (0, \infty)$, $f_*, f^* \in C^\theta_{\mathrm{loc}}([0, 1) \times [0, \infty))$. Both $f_*(t, z), f^*(t, z)$ are continuously differentiable in z, strictly increasing in $z \ge 0$ for every fixed $t \in [0, 1)$ and satisfy

$$0 < f_*(|x|, |p|) \le f(x, p) \le f^*(|x|, |p|), \quad (x, p) \in B \times R^N. \tag{3.1}$$

The main results of this paper are as follows:

Theorem 3.1. *Suppose that conditions (B_1) and (B_2) hold, f_* and f^* satisfy the conditions (A_2) and (A_3). Then, there exists a positive solution u of class $C^2(B) \cap C(\overline{B})$ for singular elliptic boundary value problems (1.5)–(1.6).*

Proof. We consider the following boundary value problems:

$$\Delta u + f^*(|x|, |\nabla u|)u^{-\lambda} = 0, \quad x \in B, \tag{3.2}$$

$$u = 0, \quad x \in \partial B; \tag{3.3}$$

$$\Delta u + f_*(|x|, 0)u^{-\lambda} = 0, \quad x \in B, \tag{3.4}$$

$$u = 0, \quad x \in \partial B. \tag{3.5}$$

Applying Lemma 2.2 to these problems, we see that problems (3.2)–(3.3) and (3.4)–(3.5), respectively, have positive radial solutions $\overline{u}(|x|)$ and $\underline{u}(|x|)$ of class $C^2_{\mathrm{loc}}(B) \cap C(\overline{B})$. Note that $f^*, f_* \in C^2_{\mathrm{loc}}$. The regular theorem (see [7, p. 109]) implies that $\overline{u}, \underline{u} \in C^{2+\theta}_{\mathrm{loc}}(B) \cap C(\overline{B})$. It is obvious that $\overline{u}, \underline{u}$ are a supersolution and a subsolution respectively of the boundary value problems (1.5)–(1.6). We next prove that $\overline{u}(|x|) \ge \underline{u}(|x|)x \in B$. Since $\overline{u} - \underline{u}$ satisfies

$$\Delta(\overline{u} - \underline{u}) + f^*(|x|, |\nabla \overline{u}|)\overline{u}^{-\lambda} - f_*(|x|, 0)\underline{u}^{-\lambda} = 0, \quad x \in B, \tag{3.6}$$

$$\overline{u} - \underline{u} = 0, \quad x \in \partial B. \tag{3.7}$$

We can change (3.6) as follows:

$$\Delta(\overline{u}-\underline{u})+f^*(|x|,|\nabla\overline{u}|)\overline{u}^{-\lambda}-f_*(|x|,0)\underline{u}^{-\lambda}+f_*(|x|,0)u^{-\lambda}-f_*(|x|,0)\overline{u}^{-\lambda}=0,\quad x\in B. \tag{3.8}$$

Condition (B_2) and (3.8) can be applied to obtain

$$\Delta(\overline{u}-\underline{u})+f_*(|x|,0)\overline{u}^{-\lambda}-f_*(|x|,0)\underline{u}^{-\lambda}\le 0,\quad x\in B. \tag{3.9}$$

Thus, we obtain

$$\Delta(\overline{u}-\underline{u})+c(x)\left(\overline{u}-\underline{u}\right)\le 0,\quad x\in B. \tag{3.10}$$

$$\begin{aligned} c(x) &= \int_0^1 \left.\frac{\partial f_*(|x|,0)\overline{u}^{-\lambda}}{\partial u}\right|_{u=t\overline{u}+(1-t)\underline{u}} \mathrm{d}t, \\ &= -\lambda f_*(|x|,0)\int_0^1 (t\overline{u}+(1-t))\overline{u}^{-(1+\lambda)}\mathrm{d}t\le 0. \end{aligned} \tag{3.11}$$

Making use of (3.10), (3.7) and the weak maximum principle of linear equation (see [7, p. 33]), we find that $\inf_B(\overline{u}-\underline{u})\ge \inf_{\partial B}(\overline{u}-\underline{u})=0$. Thus, we have

$$\overline{u}(|x|)\ge\underline{u}(|x|),\quad x\in B. \tag{3.12}$$

Let $B_n=\{x\in R^N \mid |x|<1-\frac{1}{n}\}$ for $n=2,3,\ldots$, and h be a function of class $C^{2+\theta}_{\text{loc}}(B)\cap C(\overline{B})$ satisfying $\overline{u}(|x|)\le h(x)\le\underline{u}(|x|)$ in B. Since $\overline{u}(|x|)$ and $\underline{u}(|x|)$ are a supersolution and a subsolution respectively, of boundary value problems (1.5)–(1.6), we see clearly that $\overline{u}$, $\underline{u}$ are also a supersolution and a subsolution of the following boundary value problems

$$\Delta u+f(x,\nabla u)u^{-\lambda}=0,\quad x\in B_n, \tag{3.13}$$

$$u=h(x),\quad x\in\partial B_n, \tag{3.14}$$

for each $n\ge 2$, and satisfy

$$\overline{u}(|x|)\ge\underline{u}(|x|),\quad x\in\overline{B}_n. \tag{3.15}$$

Using condition (B_1) and supersolution and a subsolution obtained as above, we find that there exists a positive $u_n\in C^{2+\theta}_{\text{loc}}\left(\overline{B}_n\right)$ $(n\ge 2)$ (see [8, p. 336]) for boundary value problems (3.13)–(3.14) and satisfy

$$\underline{u}(|x|)\le u_n(x)\le\overline{u}(|x|),\quad x\in\overline{B}_n. \tag{3.16}$$

By the standard estimation arguments for elliptic equations, we have

$$\left\|u_j\right\|_{c^{2+\theta}\overline{B}_n}\le C(n),\quad j\ge n+1, \tag{3.17}$$

where $C(n)$ is independent of [8, p. 340]. Since the injection $C^{2+\theta}(\overline{B}_n)\to C^2(\overline{B}_n)$ is compact (see [9, p. 11]), the usual diagonal argument shows that there is a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ which converges in the C^2-topology to some positive function $u_\infty\in C^2(B)$ on each compact subset of B. It is easily seen that this u_∞ vanishes identically on ∂B. Therefore, u_∞ is the desired positive solution of problems (1.5)–(1.6) of class $C^2(\overline{B})\cap C(\overline{B})$. This completes the proof of Theorem 3.1. □

4. Example

We consider the singular elliptic boundary value problem

$$\Delta u+\frac{\varphi(x)}{(1-|x|^2)^\omega}\left(k-\frac{1}{1+|\nabla u|^2}\right)u^{-\lambda}=0,\quad x\in B, \tag{4.1}$$

$$u=0,\quad x\in\partial B, \tag{4.2}$$

where $B=\{x\in R^n \mid |x|<1\}$ as above, $0<\omega<1$, $k>1$, $\lambda>0$, $\varphi\in C^\theta(\overline{B})$, $(\theta\in(0,1))$ and satisfies $0<a\le\varphi(x)\le b$, $x\in\overline{B}$, where a and b are both constants. Let

$$f(x,p)=\frac{\varphi(x)}{(1-|x|^2)^\omega}\left(k-\frac{1}{1+|p|^2}\right)u^{-\lambda},\quad (x,p)\in B\times R^N.$$

Next, we shall verify step by step that the function $f(x,p)$ satisfies the condition of Theorem 3.1.

(B_1). Since $\varphi \in C^\theta(\overline{B})$ and $\frac{1}{(1-|x|^2)^\omega}$ is continuously differentiable in B. Thus, $\frac{\varphi(x)}{(1-|x|^2)^\alpha} \in C^\theta_{\mathrm{loc}}(B)$. It is clear that $k - \frac{1}{1+|p|^2}$ is continuously differentiable in R^N and $k - \frac{1}{1+|p|^2} > 0$. Thus, $f : B \times R^N \to (0, \infty)$ and $f \in C^\theta_{\mathrm{loc}}(B \times R^N)$, $(\theta \in (0, 1))$ and f is continuously differentiable in p. For every compact region $\Omega \subset B$, let $\rho_\Omega = k \max_{x\in\Omega} \frac{\varphi(x)}{(1-|x|^2)^\omega}$, then we have

$$|f(x, p)| \le \rho_\Omega \le \rho_\Omega(1 + |p|^2), \quad x \in \Omega,\ p \in R^N.$$

(B_2). Put $\varphi_*(t) = \inf_{|x|=t} \varphi(x)$, $\varphi^*(x) = \max_{|x|=t} \varphi(x)$, $t \in [0, 1)$ and

$$f_*(t, z) = \frac{\varphi_*(t)}{(1 - t^2)^\omega} \left(k - \frac{1}{1 + z^2}\right), \quad (t, z) \in [0, 1) \times [0, \infty),$$
$$f^*(t, z) = \frac{\varphi^*(x)}{(1 - t^2)^\omega} \left(k - \frac{1}{1 + z^2}\right), \quad (t, z) \in [0, 1) \times [0, \infty).$$

Then $f_*, f^* : [0, 1) \times [0, \infty) \to (0, \infty)$. Since $\varphi \in C^\theta(\overline{B})$, $\varphi_*, \varphi^* \in C^\theta[0, 1)$ and f_*, f^* are both continuously differentiable in $z \ge 0$. Thus, $f_*, f^* \in C^\theta_{\mathrm{loc}}([0, 1) \times [0, \infty))$ and for fixed $t \in [0, 1)$, f_*, f^* are both strictly increasing in $z \ge 0$, and satisfy

$$f_*(|x|, |p|) \le f(x, p) \le f^*(|x|, |p|), \quad (x, p) \in B \times R^N.$$

Below we verify that f_*, f^* satisfy conditions (A_2) and (A_3). Note that $0 < a \le \varphi(x) \le b, x \in \overline{B}$, and thus we must only verify that

$$F(t, z) = \frac{1}{(1 - t^2)^\omega} \cdot \left[k - \frac{1}{1 + z^2}\right], \quad (t, z) \in [0, 1) \times [0, \infty)$$

satisfies conditions (A_2) and (A_3).

(A_2). Since

$$\begin{aligned} 0 &< \int_0^1 \int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, d)\mathrm{d}r\mathrm{d}s \\ &= \int_0^1 \int_0^s \left(\frac{r}{s}\right)^{N-1} \frac{1}{(1 - r^2)^\omega} \left(k - \frac{1}{1 + d^2}\right) \mathrm{d}r\mathrm{d}s \\ &< k\int_0^1 \int_0^s \left(\frac{r}{s}\right)^{N-1} \frac{\mathrm{d}r\mathrm{d}s}{(1 - r^2)^\omega} < k\int_0^1 \int_0^s \frac{\mathrm{d}r\mathrm{d}s}{(1 - r)^\omega} \\ &= \frac{k}{1 - \varpi} \int_0^1 [1 - (1 - s)^{1-\varpi}]\mathrm{d}s = \frac{k}{2 - \varpi} = M \end{aligned}$$

which holds uniformly for $d \ge 0$.

(A_3). When $0 < t < 1$, we find that

$$\begin{aligned} 0 &< \int_0^1 \int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, d)\mathrm{d}r\mathrm{d}s \le \frac{k}{1 - \varpi} \int_0^1 [1 - (1 - s)^{1-\varpi}]\mathrm{d}s \\ &= \frac{k(1 - t)}{1 - \varpi} \left[1 - \frac{(1 - t)^{1-\varpi}}{2 - \varpi}\right] < \frac{k}{1 - \varpi}(1 - t). \end{aligned}$$

Thus, for all $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon(1-\varpi)}{k}$, where $1 - \delta < t < 1$ such that

$$\int_0^1 \int_0^s \left(\frac{r}{s}\right)^{N-1} F(r, d)\mathrm{d}r\mathrm{d}s < \varepsilon$$

holds uniformly for $d \ge 0$.

From the above verification, we see that the function $f(x, p)$ in Eq. (4.1) satisfied all conditions of Theorem 3.1. By Theorem 3.1, we conclude that there exists a solution $u \in C^2(B) \cap C(\overline{B})$ of the boundary value problems (4.1)–(4.2).

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